MATH 4030 Differential Geometry
Problem Set 6
due 1/12/2017 (Fri) at 5PM

Problems

(to be handed in)

1. Show that if all the geodesics of a connected surface are plane curves, then the surface is contained in a plane or a sphere.
2. If $S \subset \mathbb{R}^3$ is a closed surface with positive Gauss curvature $K > 0$, show that any two simple closed geodesics on S must intersect.
3. Let $\Sigma \subset \mathbb{R}^3$ be a closed orientable surface which is not homeomorphic to a sphere. Prove that there are points on Σ where the Gauss curvature K is positive, negative and zero respectively.
4. Let T be a torus of revolution which can be parametrized (except along two curves) by
$$f(u,v) = ((2+\cos u)\cos v, (2+\cos u)\sin v, \sin u), \qquad 0 < u < 2\pi, 0 < v < 2\pi.$$
Prove by an explicit calculation that the total Gauss curvature is
$$\int_T K\, dA = 0.$$
5. Let $\alpha(s) : [0,L] \to \mathbb{R}^2$ be a simple closed convex plane curve p.b.a.l which is oriented in the counterclockwise direction. Let $\{T(s), N(s)\}$ be the Frenet frame of $\alpha(s)$. The curve $\beta(s) : [0,L] \to \mathbb{R}^2$ given by
$$\beta(s) := \alpha(s) - rN(s),$$
where $r > 0$ is a constant, is called a *parallel curve* to α. Show that
   (a) Length (β) = Length $(\alpha) + 2\pi r$.
   (b) Area (Ω_β) = Area $(\Omega_\alpha) + rL + \pi r^2$,

   where Ω_α, Ω_β are the regions bounded by α and β respectively.
6. Let $\alpha(s) : [0,L] \to \mathbb{R}^3$ be a closed space curve p.b.a.l. with curvature $k > 0$ and torsion τ. Denote $\{T(s), N(s), B(s)\}$ to be the Frenet frame of $\alpha(s)$. We can regard $N(s) : [0,L] \to \mathbb{S}^2$ as a curve on the unit sphere $\mathbb{S}^2$.
   (a) Show that $\|\frac{dN}{ds}\| = \sqrt{k^2+\tau^2}$ and hence $N(s) : [0,L] \to \mathbb{S}^2$ is a regular curve on $\mathbb{S}^2$.
   (b) Let $t = t(s)$ be the arc length parameter of the curve N in (a). Show that
$$\frac{dN}{dt} = \frac{1}{\sqrt{k^2+\tau^2}}(-kT - \tau B).$$

(c) Show that the geodesic curvature k_g of N as a curve in $\mathbb{S}^2$ (oriented by the outward unit normal) is given by

$$k_g = -\frac{d}{ds}\left(\tan^{-1}\frac{\tau}{k}\right)\left(\frac{dt}{ds}\right)^{-1}.$$

(d) Assume furthermore that $N : [0, L] \to \mathbb{S}^2$ is a simple closed curve. Show that the image of N divides $\mathbb{S}^2$ into two regions with the same area.

Suggested Exercises

(no need to hand in)

1. Study the geodesics on a torus of revolution.

2. Follow the steps below to give a proof of the isoperimetric inequality in the plane. Let $\alpha(s) : [0, L] \to \mathbb{R}^2$ be a simple closed plane curve p.b.a.l. Let A be the area of the region Ω bounded by the curve α.

   (a) Show that

   $$2A \leq \left(\int_0^L |\alpha(s)|^2 \, ds\right)^{\frac{1}{2}} L^{\frac{1}{2}}.$$

   (b) Prove that there exists some $s_0 \in (0, L/2)$ such that the line joining $\alpha(0)$ and $\alpha(L/2)$ is perpendicular to the line joining $\alpha(s_0)$ and $\alpha(s_0 + L/2)$.

   (c) By (b), we can assume after a translation and rotation of $\mathbb{R}^2$ that the two perpendicular lines in (b) intersect at the origin and that

   $$\frac{\alpha(L/2) - \alpha(0)}{|\alpha(L/2) - \alpha(0)|} = (1, 0) = e_1, \qquad \frac{\alpha(s_0 + L/2) - \alpha(s_0)}{|\alpha(s_0 + L/2) - \alpha(s_0)|} = (0, 1) = e_2.$$

   Prove that

   $$\int_0^L |\alpha(s)|^2 \, ds \leq \frac{L^3}{4\pi^2}$$

   using the *Wirtinger's inequality* which says that

   $$\int_a^b (f'(t))^2 \, dt \geq \frac{\pi^2}{(b-a)^2} \int_a^b (f(t))^2 \, dt \tag{1}$$

   for any smooth function $f : [a, b] \to \mathbb{R}$ such that $f(a) = f(b) = 0$. Moreover equality holds in (1) if and only if $f(t) = C \sin \frac{\pi(t-a)}{b-a}$ for some constant $C \in \mathbb{R}$.

   (d) Using the above results, prove the *isoperimetric inequality* $4\pi A \leq L^2$. What about the equality case?

3. Let $\alpha(s) : [0, L] \to \mathbb{R}^2$ be a simple closed plane curve p.b.a.l. which is contained in a closed disk of radius $r > 0$. Prove that

(a) there exists $s \in [0, L]$ such that the curvature satisfies $|k(s)| \geq 1/r$;

(b) if A is the area enclosed by the curve α, then

$$A \leq \frac{r}{2} L$$

and equality holds if and only if α is a circle of radius r.

4. Let p_0 be a pole of a unit sphere $\mathbb{S}^2$ and q, r be two points on the corresponding equator in such a way that the meridians $p_0 q$ and $p_0 r$ make an angle θ at p_0. Consider a unit vector v tangent to the meridian $p_0 q$ at p_0, and take the parallel transport of v along the closed curve made up by the meridian $p_0 q$, the parallel qr, and the meridian $r p_0$ (Fig. 4-21). Determine the angle between the final position of v after parallel transport with the initial vector v.

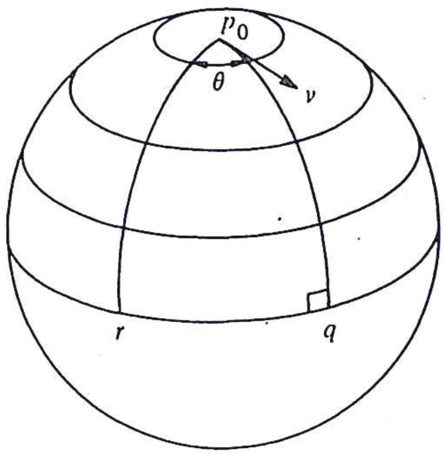

Figure 4-21